



Munich Personal RePEc Archive

A Note on Stability of Self-Consistent Equilibrium in an Asynchronous Model of Discrete-Choice with Social Interaction

Taisei Kaizoji

Graduate School of Arts and Sciences, International Christian
University

10. May 2012

Online at <http://mpra.ub.uni-muenchen.de/38730/>

MPRA Paper No. 38730, posted 22. August 2012 14:04 UTC

A Note on Stability of Self-Consistent Equilibrium in an Asynchronous Model of Discrete-Choice with Social Interaction

Taisei Kaizoji¹

Abstract

The aim of this paper is to demonstrate that dynamic paths in a model of discrete choice with social interactions, which have been developed by Brock and Durlauf (1999, 2001a, 2001b, 2006), converge some self-consistent equilibrium. To this aim, we propose an asynchronous model of discrete-choice with social interaction², in which the only individual selected cyclically is updated.

1. The model of discrete-choice with social interaction

We consider the model of discrete choice with social interaction proposed by Brock and Durlauf (1999, 2001a, 2001b, 2006). Each of I agents faces a binary choice. These choices are denoted by an indicator variable ω_i which has support $\{-1, 1\}$. Agent i makes a choice in order to maximize a utility function,

$$U(\omega_i, \mu_i^e(\omega_{-i}), \varepsilon_i(\omega_i)) = u(\omega_i) + S(\omega_i, \mu_i^e(\omega_{-i})) + \varepsilon(\omega_i) \quad (1)$$

where $\omega_{-i} = (\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_I)$ denotes the vector of choices other than that of agent i

and $\mu_i^e(\omega_{-i})$ denotes that individual's anticipations concerning the choices of other agents.

The three terms which construct the utility function are named respectively the following:

$u(\omega_i)$ is deterministic private utility, $S(\omega_i, \mu_i^e(\omega_{-i}))$ is deterministic social utility, and

$\varepsilon(\omega_i)$ is random private utility. The deterministic social utility represents a conformity effect,

¹ Taisei Kaizoji, International Christian University, 3-10-2 Osawa, Mitaka, Tokyo 181-8585 Japan.

Email: kaizoji@icu.ac.jp

² The asynchronous model is formulated in context of Neural Networks by Peterson, C. and Anderson, J. R., (1987).

$$S(\omega_i, \mu_i^e(\omega_{-i})) = -E_i \sum_{j \neq i} \frac{J_{i,j}}{2} (\omega_i - \omega_j)^2. \quad (2)$$

The term $\frac{J_{i,j}}{2}$ represents the interaction weight which relates agent i 's choice to agent j 's choice expected by agent i , and is assumed to be equal to the cross-partial derivative of the social utility function,

$$J_{i,j} = \frac{\partial^2 S(\omega_i, \mu_i^e(\omega_{-i}))}{\partial \omega_i \partial E_i(\omega_j)} \quad (3)$$

which means that the function measures the strategic complementarity (and substitutability) between agent i 's choice and the expected choices of agent j .

It is assumed that the private deterministic utility function of agent i with a linear function,

$$u(\omega_i) = h\omega_i + k \quad (4)$$

where h and k are chosen so that $h+k = u(1)$ and $-h+k = u(-1)$.

The random private utility $\varepsilon(\omega_i)$ is independent and extreme value distributed both within and across agents³. This means that the difference between the unobservable components is logistically distributed

$$\chi(\varepsilon(-1) - \varepsilon(1) \leq z) = \frac{1}{1 + \exp(-\beta z)}; \quad \beta \geq 0. \text{ d}, \quad (5)$$

where $\chi(\cdot)$ denotes probability measures.

Under the above assumptions, agent i 's choice will obey the probability

$$\Pr(\omega_i) = \frac{\exp\left(\beta(u(\omega_i) + \sum_{i \neq j} J_{i,j} \omega_i E_i(\omega_j))\right)}{\sum_{v_i \in \{-1,1\}} \exp\left(\beta(u(v_i) + \sum_{i \neq j} J_{i,j} \omega_i E_i(\omega_j))\right)} \quad (6).$$

The expected value of agent i 's choice, which is conditional on his anticipations concerning the

³ See McFaden (1984).

behavior of others, can be written as

$$E(\omega_i) = \tanh(\beta h_i + \beta \sum_{j \neq i} J_{i,j} E(\omega_j)) \quad (7)$$

To close the model, Brock and Durlauf (2000, 2001) considers a case of this model occurs when all of the agents possess rational expectations, i.e.

$$E_i(\omega_j) = E(\omega_j). \quad (8)$$

Under the assumption of expectation formations, the subjective expectations can be replaced with their objective expectations, i.e.

$$E(\omega_i) = \tanh(\beta h_i + \beta \sum_{j \neq i} J_{i,j} E(\omega_j)) \quad (i = 1, 2, \dots, I). \quad (9)$$

These equations represent a continuous mapping of $C = (-1, 1)^I \rightarrow C$. Therefore, it is clear from Brouwer's fixed point theorem that there is at least one fixed point solution, which implies Theorem 1. (see Brock and Durlauf (1999, 2001a, 2001b)).

Theorem 1. Existence of self-consistent equilibria

There exists at least one set of self-consistent equilibrium consistent with the binary choice model with interactions as specified by equation (9).

2. Dynamics

In order to consider the dynamic stability of the self-consistent equilibrium in the discrete-choice model from the point of view of dynamical system theory, we consider an asynchronous discrete-choice model in which the only individual selected cyclically is updated.

2.1. The synchronous discrete-choice model.

For simplicity of the description of the asynchronous discrete-choice model, let a mapping

$f : C = (-1, 1)^n \rightarrow C$ be defined by

$$f_i(x) = \tanh \left(\beta \left(\sum_{j=1}^n J_{i,j} x_j(t) + h_i \right) \right) \quad (i = 1, \dots, n) \quad (10)$$

where $x_i = E(\omega_i)$, $x = (x_1, x_2, \dots, x_n)$, and, $f(x) = {}^t (f_1(x), \dots, f_n(x))$.

In order to consider the asynchronous discrete-choice model from the point of view of dynamical system theory, we define a mapping $f : C \rightarrow C$ as follows.

$$\tilde{f}_1(x) = f_1(x_1, x_2, \dots, x_n)$$

$$\tilde{f}_2(x) = f_2(\tilde{f}_1(x), x_2, \dots, x_n)$$

...

...

...

$$\tilde{f}_n(x) = f_n(\tilde{f}_1(x), \tilde{f}_2(x), \dots, \tilde{f}_{n-1}(x), x_n)$$

and

$$\tilde{f}(x) = {}^t (\tilde{f}_1(x), \dots, \tilde{f}_n(x)).$$

Then, the asynchronous discrete-choice model can be represented by

$$x(t+1) = \tilde{f}(x(t)) \quad (t = 0, 1, 2, \dots). \quad (11)$$

2.2. Stability of self-consistent equilibrium

We assume that

$$J_{i,j} = J_{j,i}. \quad (12)$$

That is, the interactions which are driven by expectations of the behavior of others are symmetry, and no self-connection (i.e. $J_{i,i} = 0$).

Under these conditions, the following lemma 1 and Theorem 2 are demonstrated. The proofs are due to essentially Fleisher (1988) and Kurita and Funahashi (1996).

Lemma 1. *Assume that a matrix J of pairwise interactions between each individual choice and the expected choices of others are symmetric, that is, $J_{i,j} = J_{j,i}$, and no self-connection (i.e. $J_{i,i} = 0$). We define a Liapunov function,*

$$V(x) = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n J_{i,j} x_i x_j - \sum_{i=1}^n h_i x_i + \frac{1}{\beta} \sum_{i=1}^n \int_0^{x_i} \sigma^{-1}(x) dx \quad (13)$$

where $\sigma(x) = \tanh(x)$. Then, the Liapunov function () of the synchronous discrete-choice model with the weight matrix $J_{i,j}$ decreases monotonically. If $x' = \tilde{f}(x)$ and $x' \neq x$, then $V(x') < V(x)$, when $\tilde{f}: C \rightarrow C$ is a mapping which defines the asynchronous discrete-choice model.

Theorem 2 (Dynamic stability). *The state of the asynchronous discrete-choice model converges to some self-consistent equilibrium.*

3. Acknowledgement

This research was supported in part by a grant from NOMURA Foundation.

4. Appendix

The proof of Lemma 1 is given by Lemma 9 in Kurita and Funahashi (1996).

Proof of lemma 1 (Kurita and Funahashi (1996)). First, we prove the first assertion. We remember that

$$V(x) = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n J_{i,j} x_i x_j - \sum_{i=1}^n h_i x_i + \frac{1}{\beta} \sum_{i=1}^n \int_0^{x_i} \sigma^{-1}(u) du.$$

We set

$$g(x_i) = \frac{1}{\beta} \int_0^{x_i} \sigma^{-1}(u) du.$$

Suppose that x_i changed to $x'_k = x_k + \Delta x_k$, the resulting change in V is given by

$$\Delta V = -\Delta x_k \left\{ \sum_{i=1}^n J_{k,j} x_j + h_k - \frac{g(x_k + \Delta x_k) - g(x_k)}{\Delta x_k} \right\}.$$

Applying the intermediate value theorem to the function $g(x)$, we get

$$\begin{aligned} \Delta V &= -\Delta x_k \left\{ \sum_{i=1}^n J_{k,j} x_j + h_k - g'_k(\xi) \right\} \\ &= -\frac{1}{\beta} \Delta x_k \left\{ \sigma^{-1}(x_k + \Delta x_k) - \sigma^{-1}(\xi) \right\}, \end{aligned}$$

where ξ is a point between x_k and $x_k + \Delta x_k$. Between both $g(x)$ and $\sigma^{-1}(x)$ are strictly increasing functions, if $\Delta x_k > 0$ then $x_k < \xi < x_k + \Delta x_k$ and this implies

$$\sigma^{-1}(\xi) < \sigma^{-1}(x_k + \Delta x_k)$$

and hence $\Delta V < 0$. A similar argument holds for the case $\Delta x_k < 0$. Of course $\Delta x_k = 0$ implies $\Delta V = 0$.

Second, we prove the last part of the lemma. From the first assertion, there is no periodic point in the asynchronous model. Hence if $x' = \tilde{f}(x)$ and $x' \neq x$, then $V(x') < V(x)$.

Q.E.D.

The proof of Theorem 2 is given by Theorem 5 in Kurita and Funahashi (1996).

Proof of Theorem 2 (Kurita and Funahashi (1996)).

We take any initial point $x \in C = (-1, 1)^n$. We remark that $\bar{f}(C)$ is a relatively compact subset of C because $\sigma: R \rightarrow (-1, 1)$ is a bounded function. We consider the ω -limit set of x :

$$\omega_{\bar{f}}(x) = \left\{ y \in C \mid \text{there is a sequence } n_i \rightarrow \infty \text{ such that } \bar{f}^{n_i}(x) \rightarrow y \right\}.$$

As $\bar{f}(C)$ is a relatively compact subset of C , $\omega_{\bar{f}}(x)$ is a nonempty relatively compact subset of C . Suppose that $\omega_{\bar{f}}(x)$ contains at least two points p and q ($p \neq q$). From the definition and the above lemma, we easily see that the Liapunov function $V(x)$ is constant on $\omega_{\bar{f}}(x)$.

Hence, for any $\varepsilon > 0$, there are two integers N_1 and N_2 such that $N_1 > N_2$,

$$\left| \bar{f}^{N_1}(x) - p \right| < \varepsilon, \text{ and } \left| \bar{f}^{N_2}(x) - q \right| < \varepsilon.$$

Because $V(\bar{f}^{N_2}(x)) < V(\bar{f}^{N_1}(x))$, this implies $V(q) < V(p)$ from the above lemma and this is a contradiction. Therefore $\omega_{\bar{f}}(x)$ contains only one point. Q.E.D.

5. References

Brock, W. and S. Durlauf, 1999, "A Formal Model of Theory Choice in Science," *Economic Theory* 14, pp.113-130 .

Brock, W., and S. Durlauf, 2001a, "Discrete Choice with Social Interactions," *Review of Economic Studies*, 68, 2, pp.235-260.

Brock, W. and Durlauf, S., 2001b, "Interactions-Based Models," in *Handbook of Econometrics*, Vol. 5, Chapter 54, pp. 3297-3380, Edited by J.J. Heckman and E. Leamer, Elsevier Science B.V.

Brock, W. and S. Durlauf, 2006, "Multinomial choice model with social interactions," *The Economy as an Evolving Complex System*, III, pp. 175-206, edited by L.E. Blume and S.N. Durlauf, The Oxford University Press, Inc., New York NY.

Fleisher, M., 1988, "The Hopfield model with multi-level neurons", In D. Z. Anderson (Ed.), *Neural Information Processing Systems, Denver, CO*, pp. 278-289, American Institute of Physics: New York.

Kurita, N. and Funahashi, K., 1996, "On the Hopfield neural networks and mean field theory," *Neural Networks*, Vol. 9, no. 9, pp. 1531-1540.

McFadden, D. 1984, "Econometric Analysis of Qualitative Response Models," in Z. Griliches and M. Intriligator (eds.), *Handbook of Econometrics: Volume II* (Amsterdam: North-Holland).

Peterson, C. and Anderson, J. R., 1987, "A mean field theory learning algorithm for neural networks," *Complex Systems*, Vol. 1, pp. 995-1019,.